One purpose of this bibliography is to guide the reader to other sources, but the most important function it can serve is to indicate the variety of mathematical reading available. Consequently, there is an attempt to achieve diversity, but no pretense of being complete. The present plethora of mathematics books would make such an undertaking almost hopeless in any case, and since I have tried to encourage independent reading, the more standard a text, the less likely it is to appear here. In some cases, this philosophy may seem to have been carried to extremes, as some entries in the list cannot be read by a student just finishing a first course of calculus until several years have elapsed. Nevertheless, there are many selections which can be read now, and I can't believe that it hurts to have some idea of what lies ahead.

For most references, only the title and author have been given, since so many of these books have gone through numerous editions and printings, often having gone out of print at some point only to be resurrected later on by a different publisher (often as an inexpensive paperback by the redoubtable Dover Publications or by the Mathematical Association of America). More exact information really isn't necessary, since it is now so easy to search for books on-line at Amazon.com and other sites.

† is used to indicate books whose availability, either new or used, is problematical. Author and title searches may turn up other intriguing books by the same author, or other books with similar titles. In addition, many of these books will still be found in well-stocked academic libraries, perhaps the best place of all to search; despite the convenience of the internet, nothing matches the experience of an actual (as opposed to a virtual) library, with books stacked according to subject, awaiting serendipitous discovery.

One of the most elementary unproved theorems mentioned in this book is the "Fundamental Theorem of Arithmetic", that every natural number can be written as a product of primes in only one way. This follows from the basic fact alluded to on page 444, a proof of which will be found near the beginning of almost any book on elementary number theory. Few books have won so enthusiastic an audience as

[1] *An Introduction to the Theory of Numbers*, by G. H. Hardy and E. M. Wright.

Two other recommended books are

† [2] *A Selection of Problems in the Theory of Numbers*, by W. Sierpinski.
[3] *Three Pearls of Number Theory*, by A. Khinchin.

The Fundamental Theorem also applies in more general algebraic settings, see references [33] and [34].

The subject of irrational numbers straddles the fields of number theory and analysis. An excellent introduction will be found in

[4] *Irrational Numbers*, by I. M. Niven.

Together with many historical notes, there are references to some fairly elementary articles in journals. There is also a proof that $\pi$ is transcendental (see also [59]) and, finally, a proof of the "Gelfond-Schneider theorem": If $a$ and $b$ are algebraic, with $a \neq 0$ or 1, and $b$ is irrational, then $a^b$ is transcendental.

All the books listed so far begin with natural numbers, but whenever necessary take for granted the irrational numbers, not to mention the integers and rational numbers. Several books present a construction of the rational numbers from the natural numbers, but one of the most lucid treatments is still to be found in

[5] *Foundations of Analysis*, by E. Landau.

While many mathematicians are content to accept the natural numbers as a natural starting point, numbers can be defined in terms of sets, the most basic starting point of all. A charming exposition of set theory can be found in a sophisticated little book called

[6] *Naive Set Theory*, by P. R. Halmos.

Another very good introduction is

[7] *Theory of Sets*, by E. Kamke.

Perhaps it is necessary to assure some victims of the "new math" that set theory does have some mathematical content (in fact, some very deep theorems). Using these deep results, Kamke proves that there is a discontinuous function $f$ such that $f(x+y) = f(x) + f(y)$ for all $x$ and $y$.

Inequalities, which were treated as an elementary topic in Chapters 1 and 2, actually form a specialized field. A good elementary introduction is provided by

[8] *Analytic Inequalities*, by N. Kazarinoff.

Twelve different proofs that the geometric mean is less than or equal to the arithmetic mean, each based on a different principle, can be found in the beginning of the more advanced book

[9] *An Introduction to Inequalities*, by E. Beckenbach and R. Bellman.

The classic work on inequalities is

[10] *Inequalities*, by G. H. Hardy, J. E. Littlewood, and G. Polya.

Each of the authors of this triple collaboration has provided his own contribution to the sparse literature about the nature of mathematical thinking, written from a mathematician's point of view. My favorite is

[11] *A Mathematician's Apology*, by G. H. Hardy.

Littlewood's anecdotal selections are entitled

† [12] *A Mathematician's Miscellany*, by J. E. Littlewood.

Polya's contribution is pedagogy at the highest level:

[13] *Mathematics and Plausible Reasoning* (Vol. I: *Induction and Analogy in Mathematics;* † Vol. II: *Patterns of Plausible Inference*), by G. Polya.

Geometry is the other main field which can be considered as background for calculus. Though Euclid's *Elements* is still a masterful mathematical work, greater perspective is supplied by some more modern texts, which examine foundational questions, non-Euclidean geometry, the role of the "Archimedean axiom" in geometry, and further results from "classical geometry". Of the following three books, the first, listed in previous editions of this book, has probably been supplanted by the later ones, which cover some more advanced material, and perhaps require a little more sophistication on the part of the reader.

† [14] *Elementary Geometry from an Advanced Standpoint*, by E. Moise.
[15] *Euclidean and Non-Euclidean Geometries*, by M. J. Greenberg.
[16] *Geometry: Euclid and Beyond*, by R. Hartshorne.

In addition, all sorts of fascinating geometric things can be found in

[17] *Introduction to Geometry*, by H. S. Coxeter.

Almost all treatments of geometry at least mention convexity, which forms another specialized topic. I cannot imagine a better introduction to convexity, or a better mathematical experience in general, than reading and working through

† [18] *Convex Figures*, by I. M. Yaglom and W. G. Boltyanskii.

This book contains a carefully arranged sequence of definitions and *statements* of theorems, whose proofs are to be supplied by the reader (worked-out proofs are supplied in the back of the book). Its current unavailability is perhaps a testament to the lack of interest in working through exercises, which might also apply to another geometry book modeled on the same principle:

† [19] *Combinatorial Geometry in the Plane*, by H. Hadwiger and H. Debrunner.

Along with these two out-of-the-ordinary books, I might mention an extremely valuable little book, also of a specialized sort,

[20] *Counterexamples in Analysis*, by B. Gelbaum and J. Olmsted.

Many of the examples in this book come from more advanced topics in analysis, but quite a few can be appreciated by someone who knows calculus.

Of the infinitude of calculus books, two are considered classics:

[21] *A Course of Pure Mathematics*, by G. H. Hardy.
[22] *Differential and Integral Calculus* (two volumes), by R. Courant.

Courant is especially strong on applications to physics. There is also a more modern update

[23] *Introduction to Calculus and Analysis*, by R. Courant and F. John.

(†*Problems in Calculus and Analysis* by Albert A. Blank is a sort of companion to Volume 1, including additional exercises and problems, as well as answers and solutions to many of the problems in that volume.)

Speaking of applications to physics, an elegant exposition of the material in Chapter 17, together with much further discussion, can be found in the article

[24] *On the geometry of the Kepler problem*, by John Milnor; in *The American Mathematical Monthly*, Volume 90 (1983), pp. 353–365.

(In this paper the curve $c'$ of Chapter 17 is denoted by $\mathbf{v}$, and the derivative of the important composition $\mathbf{v} \circ \theta^{-1}$ (page 334) is introduced quite off-handedly as $d\mathbf{v}/d\theta$.) A "straight-forward" derivation of Kepler's laws, together with numerous references, can be found in another article in this same journal,

[25] *The mathematical relationship between Kepler's laws and Newton's laws*, by Andrew T. Hyman; in *The American Mathematical Monthly*, Volume 100 (1993), pp. 932–936.

The later parts of Volume I of Courant contain material usually found in advanced calculus, including differential equations and Fourier series. An introduction to Fourier series (requiring a little advanced calculus) will also be found in

[26] *An Introduction to Fourier Series and Integrals*, by R. Seeley.

The second volume of Courant (advanced calculus in earnest) contains additional material on differential equations, as well as an introduction to the calculus of variations. A widely admired book on differential equations is

[27] *Lectures on Ordinary Differential Equations*, by W. Hurewicz.

A good example of new approaches and new topics is provided by

[28] *Differential Equations, Dynamical Systems, and An Introduction to Chaos*, by M. Hirsch, S. Smale, and R. L. Devaney.

I will bypass the more or less standard advanced calculus books (which can easily be found by the reader) since nowadays the presentation of advanced calculus for mathematics students is based upon linear algebra. One of the first treatments of advanced calculus using linear algebra is the very nice book

† [29] *Calculus of Vector Functions*, by R. H. Crowell and R. E. Williamson.

More recent books to be recommended are

[30] *Advanced Calculus of Several Variables*, by C. H. Edwards, Jr.
[31] *Multivariable Mathematics*, by T. Shifrin.

And of course I am still partial to an older text

[32] *Calculus on Manifolds*, by M. Spivak.

There are three other topics which are somewhat out of place in this bibliography because they are gradually becoming established as part of a standard undergraduate curriculum. The purposeful study of fields and related systems is the domain of "algebra." Two excellent texts are

[33] *Algebra*, by Michael Artin.
[34] *Abstract Algebra*, by D. Dummit and R. Foote.

For "complex analysis", the promised land of Chapter 27, the classical text is

[35] *Complex Analysis*, by L. Ahlfors.

Rather revolutionary when it was first published, it might now be considered somewhat old-fashioned, and you might prefer the second in a series of books (3 and counting) that have appeared more recently:

[36] *Fourier Analysis: An Introduction*, by E. Stein and R. Shakarchi.
[37] *Complex Analysis*, by E. Stein and R. Shakarchi.
[38] *Real Analysis*, by E. Stein and R. Shakarchi.

And, since the topic of "real analysis" [high-octane Calculus] has been broached, two classics should be mentioned. The first, affectionately known as "baby Rudin", was the source of several problems that appear in this book.

[39] *Principles of Mathematical Analysis*, by W. Rudin.
[40] *Functional Analysis*, by W. Rudin.

The subject of "topology" has not been mentioned before, but it has really been in the background of many discussions, since it is the natural generalization of the ideas about limits and continuity which play such a prominent role in Part II of this book. The standard text is now

[41] *Topology*, by J. R. Munkres.

For the related field of "differential topology", see

[42] *Differential Topology*, by V. Guillemin and A. Pollack.

The next few topics, ranging from elementary to very difficult, are included in this bibliography because they have been alluded to in the text. The gamma function has an elegant little book devoted entirely to its properties, most of them proved by using the theorem of Bohr and Mollerup which was mentioned in Problem 19-40:

† [43] *The Gamma Function*, by E. Artin

The gamma function is only one of several important improper integrals in mathematics. In particular, the calculation of $\int_0^\infty e^{-x^2}\,dx$ (see Problem 19.42) is important in probability theory, where the "normal distribution function"

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, dy$$

plays a fundamental role. A classic book on probability theory is

[44] *An Introduction to Probability Theory and Its Applications*, by W. Feller.

The impossibility of integrating certain functions in elementary terms (among them $f(x) = e^{-x^2}$) is a fairly esoteric topic. A discussion of the possibilities of integrating in elementary terms, with an outline of the impossibility proofs, and references to the original papers of Liouville, will be found in

[45] *The Integration of Functions of a Single Variable*, by G. H. Hardy.

The basically algebraic ideas behind the arguments were made much clearer over a hundred years after Liouville's work, in the paper

[46] *On Liousville's Theorem on functions with elementary integrals*, by M. Rosenlicht; in *Pacific Journal of Mathematics*, Volume 24, No. 1 (1968), pp. 153–161. (Also available on-line: go to `projecteuclid.org` and search for `Rosenlicht`.)

For a good overview of the subject, and some more recent developments, see

[47] *Integration in finite terms: the Liousville theory*, by T. Kasper; in *Mathematics Magazine*, Volume 53, No. 4 (1980), pp. 195–201.

Reference [46] makes use of the notions of "differential algebra", a field in which a related but seemingly more difficult problem had been solved earlier: There are simple differential equations ($y'' + xy = 0$ is a specific example) whose solutions cannot be expressed even in terms of indefinite integrals of elementary functions. This fact is proved on page 43 of the (60-page) book:

† [48] *An Introduction to Differential Algebra*, by I. Kaplansky

A few words should also be said in defense of the process of integrating in elementary terms, which many mathematicians look upon as an art (unlike differentiation, which is merely a skill). You are probably already aware that the process of integration can be expedited by tables of indefinite integrals. There are several books containing extensive tables of integrals (and also tables of series and products), but for most integrations it suffices to consult one of the fairly extensive tables of indefinite integrals that are available on-line, for example, at `sosmath.com`, and at `wikipedia.org`, with its ever-expanding source of generally definitive entries for mathematics and physics.

The remaining references are of a somewhat different sort. They fall into three categories, of which the first is historical.

For the history of calculus itself, an excellent comprehensive source, filled with detailed explicit examples, rather than generalized descriptions, is

[49] *The Historical Development of Calculus*, by C. H. Edwards, Jr.

Some historical remarks, and an attempt to incorporate them into the teaching of calculus, will be found in

[50] *The Calculus: A Genetic Approach*, by O. Toeplitz.

An admirable textbook on the history of mathematics in general is

[51] *An Introduction to the History of Mathematics*, by H. Eves.

As might be inferred from the quotation on page 38, the basic idea for constructing the real numbers is derived from Dedekind, whose contributions can be found in

[52] *Essays on the Theory of Numbers*, by R. Dedekind.

The most important notions of set theory, especially the proper treatment of infinite numbers, were first introduced by Cantor, whose work is reproduced in

[53] *Contributions to the Founding of the Theory of Transfinite Numbers*, by G. Cantor.

The letter of H. A. Schwarz referred to in Problem 11-69 will be found in

† [54] *Ways of Thought of Great Mathematicians*, by H. Meschkowski.

Finally, a great deal of interesting historical material may also be found on-line at the site `www-groups.dcs.st-and.ac.uk/~history/`

The second category in this final group of books might be described as "popularizations." There are a surprisingly large number of first-rate ones by real mathematicians:

[55] *What is Mathematics?*, by R. Courant and H. Robbins.
[56] *Geometry and the Imagination*, by D. Hilbert and S. Cohn-Vossen.
[57] *The Enjoyment of Mathematics*, by H. Rademacher and O. Toeplitz.
† [58] *Famous Problems of Mathematics*, by H. Tietze; Graylock Press, 1965.

One of the most renowned "popularizations" is especially concerned with the teaching of mathematics:

[59] *Elementary Mathematics from an Advanced Standpoint*, by F. Klein (vol. 1: *Arithmetic, Algebra, Analysis*; vol. 2: *Geometry*); Dover, 1948.

Volume 1 contains a proof of the transcendence of $\pi$ which, although not so elementary as the one in [4], is a direct analogue of the proof that $e$ is transcendental, replacing integrals with complex line integrals. It can be read as soon as the basic facts about complex analysis are known.

The third category is the very opposite extreme—original papers. The difficulties encountered here are formidable, and I have only had the courage to list one such paper, the source of the quotation for Part IV. It is not even in English, although you do have a choice of foreign languages. The article in the original French is in

[60] *Oeuvres Complètes d'Abel.*

It first appeared in a German translation in the *Journal für die reine und angewandte Mathematik*, Volume 1, 1826. To compound the difficulties, these references will usually be available only in university libraries. Yet the study of this paper will probably be as valuable as any other reading mentioned here. The reason is suggested by a remark of Abel himself, who attributed his profound knowledge of mathematics to the fact that he read the masters, rather than the pupils.